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The Mathematical Theory of Diffusion and Reaction in Enzymes Immoblized Artificial Membrane. The Theory of the Non-Steady State

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Abstract In this paper, mathematical model pertaining to the decomposition of enzyme–substrate complex in an artificial membrane is discussed. Here the transport through liquid membrane phases is considered. The model involves the system of non-linear reaction diffusion equations. The non-linear terms in this model are related to Michaelis–Menten reaction scheme. Approximate analytical expressions for the concentrations of substrate and product have been derived by solving the system of non-linear reaction diffusion equations using new approach of homotopy perturbation method for all values of Michaelis–Menten constant, diffusion coefficient, and rate constant. Approximate flux expression for substrate and product for non-steady-state conditions are also reported. A comparison of the analytical approximation and numerical simulation is also presented. The results obtained in this work are valid for the entire solution domain.

Keywords Membrane · Immobilized enzyme · Mathematical modeling · Non-linear equations · Homotopy perturbation method

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List of Symbols

E	Enzyme (mol cm^{-3})
S	Substrate (mol cm^{-3})
P	Product (mol cm^{-3})
k_i	Rate constants (s^{-1})
V	Maximum rate (m s^{-1})
K	Michaelis–Menten constant (mol cm^{-3})
D_s	Diffusion coefficients of substrate (cm^2 s^{-1})
D_p	Diffusion coefficients of product (cm^2 s^{-1})
$J_{S_{1m}}$	Flux expression for the substrate (Nm^2 c^{-1})
$J_{P_{1m}}$	Flux expression for the product (Nm^2 c^{-1})

Introduction

Immobilized enzymes have assumed great importance in both theoretical and applied work. Enzymes immobilization can be defined as the attachment of free or soluble enzymes to different types of supports resulting in reduction or loss of mobility of the enzymes. Immobilized enzymes have been widely used in the processing of variety of products. New strategies are continuously emerging for the formation of diverse immobilized enzymes having superior efficiency and usage. Immobilized enzymes have biomedical and industrial applications and for this reason, this area has continued to develop into an ever-expanding and multidisciplinary field during the last couple of decades. Khan and Alzohair (2010) present recent developments and used of immobilized enzymes in different fields such as in medicine, biosensor, antibiotic production, drug metabolism, food industry, biodiesel production, bioremediation, etc.

Biosensors have wide applications including biomarker detection for medical diagnostics and pathogen and toxin detection in food and water (Leung and Shankar 2007). The application of biosensors based on glucose oxidase immobilized by electropolymerization for heavy metal determination has been described (Ivanov et al. 2003a, b; Malitesta and Guascito 2005). Competition with well-established, fine-tuned chemical processes for antibiotics production is a major challenge for the industrial implementation of the enzyme synthesis of biologically important antibiotics such as β-lactam. (Giordano et al. 2006; Sio and Ouax 2004; Maladkar 1994; Kurochkina and Nys 1999). Immobilized enzymes are of great value in the processing of food samples and its analysis. The extent of lactose hydrolysis whey processing, skimmed milk production, etc. has been greatly enhanced using respective enzymes as immobilized forms (Carpio et al. 2000; Oh 2007; Gangadharan et al. 2009). Biodiesel has gained importance in the recent past for its ability to replace fossil fuels which are likely to run out within a century (Iso et al. 2001; Antolin et al. 2002; Tiwari et al. 2007; Dizge and Keskinler 2008; Jegannathan et al. 2008; Yagiz et al. 2007; Canakci and Gerpen 2003). Large number of (>100,000) commercially available dyes with over 7×10^5 ton of dyestuff are produced annually worldwide and used extensively in textile, dyeing, and printing industry (Akhtar et al. 2005a, b; Khan and Husain 2007a, b).

Lilly and Hornb (1966) presented an equation accounting for diffusion and electrical potential gradients in a Nernst-type of diffusion layer. Katchalshi and coworkers (1968) developed the model for substrate and product distribution in membranes containing enzymes (Goldman et al. 1968). Sundaram et al. (1970) derived the mass balance equations describing the kinetics of reaction in an enzyme-containing membrane immersed in a substrate solution. Mathematical model and mass balance equations describing steady-state catalysis by an enzyme immobilized in spherical particles have been reported (Sundaram et al. 1970; Kasche et al. 1971).

An analysis of reaction diffusion in a carrier-mediated transport process through the membrane is presented (Ganesan et al. 2013). Rajendran and Bieniasz (2012) analyzed the theoretical model describing the process of reaction and diffusion in glucose-responsive composite membranes, previously described by Abdekhodaie and Wu (2009). Blaedel and Kissel (1972) have derived the steady-state fluxes of substrate and product through a membrane in simple system. However, to the best our knowledge there was the no rigorous analytical expression corresponding to the concentration and fluxes through the membrane for non-steady-state conditions reported. In this paper, the approximate analytical expressions of the substrate or product concentrations and steady-state fluxes of substrate and product through a membrane are derived. Transport through the liquid and membrane phases is considered.

Formulation of the Problem

Figure 1 represents the reaction scheme for the decomposition of enzyme–substrate complex in the membrane film. The reaction occurring within the film can be rewritten as follows:

$$E + S \underset{k_2}{\overset{k_1}{\leftrightarrow}} ES \xrightarrow{k_3} E + nP, \tag{1}$$

where E, S, P and ES represent enzyme, substrate, product, and enzyme–substrate complex, respectively. k_i's are rate constants. The rate of reaction can be measured by means of Michaelis–Menten kinetics as $v = VS/(K + S)$. V is the maximum rate and K is the Michaelis–Menten constant. Using this Michaelis–Menten kinetic, the mass balance equation for substrate and product can be written as follows (Blaedel and Kissel 1972):

$$\frac{\partial S}{\partial t} = D_S \frac{\partial^2 S}{\partial x^2} - \frac{VS}{K + S} \tag{2}$$

$$\frac{\partial P}{\partial t} = D_P \frac{\partial^2 P}{\partial x^2} + \frac{VS}{K + S}, \tag{3}$$

where D_S and D_P are the diffusion coefficients of substrate and product, respectively. The boundary conditions are

$$\begin{aligned} &t = 0, \quad S = S_i, \quad P = 0 \\ &x = 0, \quad S = S_{1m}, \quad P = P_{1m} \; . \\ &x = \bar{x}, \quad S = S_{2m}, \quad P = P_{2m} \end{aligned} \tag{4}$$

The fluxes for the substrate and product are given by

$$J_{S_{1m}} = -D_S \left.\frac{dS}{dx}\right|_{x=0} \tag{5}$$

$$J_{P_{1m}} = -D_P \left.\frac{dP}{dx}\right|_{x=0}. \tag{6}$$

General Analytical Expression of Concentration of Substrate and Product Under Non-Steady Condition Using New Approach to Homotopy Perturbation Method

Recently, many authors have applied the homotopy perturbation method to various problems and demonstrated the efficiency of the homotopy perturbation method for

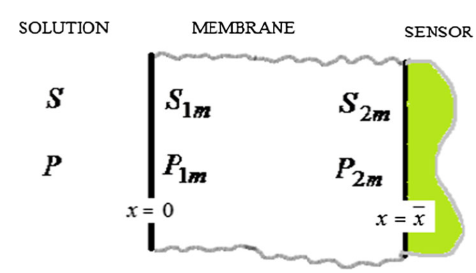

Fig. 1 Schematic representation of solution–membrane system

handling non-linear structures and solving various physics and engineering problems (Li and Liu 2006; Mousa et al. 2008; He 1999). This method is a combination of homotopy in topology and classic perturbation techniques. He used the HPM to solve the Lighthill equation (He 2003a, b), the duffing equation (He 2006), and Blasius equation (He et al. 2010). The idea has been used to solve non-linear boundary value problems, integral equations, and many other problems (He 2000; Ganji et al. 2008). The HPM is unique in its applicability, accuracy, and efficiency. The HPM uses the imbedding parameter P as a small parameter and only a few iterations are needed to search for an asymptotic solution.

Here we can assume that $D_s = D_p = D$.

Using the new approach to homotopy perturbation method and Laplace transform method, the analytical expressions of non-steady-state concentrations of substrate and product (Appendix 1) can be obtained as follows:

$$S(x,t) = \frac{S_{2m}\sinh\sqrt{A/D}x - S_{1m}\sinh\sqrt{A/D}(x-\bar{x})}{\sinh\sqrt{A/D}\bar{x}} + 2\pi D\sum_{n=0}^{\infty}\frac{e^{-(n^2\pi^2D+A\bar{x}^2)t}n\ (-1)^n\{S_{2m}\sin(n\pi x/\bar{x}) - S_{1m}\sin[n\pi(x-\bar{x})/\bar{x}]\}}{(n^2\pi^2D+A\bar{x}^2)} + \frac{2}{\pi}S_i\sum_{n=0}^{\infty}\frac{e^{-(n^2\pi^2D+A\bar{x}^2)t}(-1)^n\{\sin(n\pi x/\bar{x}) - \sin[n\pi(x-\bar{x})/\bar{x}]\}}{n}. \tag{7}$$

Using the relation between S and P (Appendix 2), we can obtain the concentration of product as follows:

$$P(x,t) = (S_{1m}+P_{1m}) + [(S_{2m}+P_{2m}) - (S_{1m}+P_{1m})]\frac{x}{\bar{x}} + \frac{2}{\pi}\sum\left[\frac{(S_{2m}+P_{2m})\cos n\pi - (S_{1m}-P_{1m})}{n}\right]\sin\left(\frac{n\pi x}{\bar{x}}\right)e^{\frac{-Dn^2\pi^2t}{\bar{x}^2}} + 2S_i\sum_{n=0}^{\infty}\sin\left(\frac{n\pi x}{\bar{x}}\right)e^{\frac{-Dn^2\pi^2t}{\bar{x}^2}}\frac{[1-(-1)^n]}{n\pi} - S(x,t). \tag{8}$$

From the above two equations, the substrate and product fluxes can be derived as

$$J_{S_{1m}} = D\left[\frac{S_{1m}\sqrt{A/D}\cosh(\sqrt{A/D}\ \bar{x}) - S_{2m}\sqrt{A/D}}{\sinh(\sqrt{A/D}\ \bar{x})}\right] - \frac{2\pi^2D^2}{\bar{x}}\sum_{n=1}^{\infty}\frac{e^{-(n^2\pi^2D+A\ \bar{x}^2)}n^2(-1)^n[S_{2m}-S_{1m}\cos\ (n\pi)]}{n^2\pi^2D+A\ \bar{x}^2} - \frac{2DS_i}{\bar{x}}\sum_{n=1}^{\infty}e^{-(n^2\pi^2D+A\ \bar{x}^2)}(-1)^n[1-\cos(n\pi)] \tag{9}$$

$$J_{P_{1m}} = D\left[\frac{(S_{1m}+P_{1m}) - (S_{2m}+P_{2m})}{\bar{x}}\right] - \frac{2D}{\bar{x}}\sum_{n=1}^{\infty}(S_{2m}+P_{2m})\cos(n\pi) - (S_{1m}+P_{1m})e^{-\frac{Dn^2\pi^2t}{\bar{x}^2}} - \frac{2DS_i}{\bar{x}}\sum e^{-\frac{Dn^2\pi^2t}{\bar{x}^2}}[1-(-1)^n] + D\left[\frac{S_{1m}\sqrt{A/D}\cosh(\sqrt{A/D}\ \bar{x}) - S_{2m}\sqrt{A/D}}{\sinh(\sqrt{A/D}\ \bar{x})}\right] + \frac{2\pi^2D^2}{\bar{x}}\sum_{n=1}^{\infty}\frac{e^{-(n^2\pi^2D+A\bar{x}^2)}n^2(-1)^n[S_{2m}-S_{1m}\cos(n\pi)]}{n^2\pi^2D+A\bar{x}^2} + \frac{2DS_i}{\bar{x}}\sum_{n=1}^{\infty}e^{-(n^2\pi^2D+A\bar{x}^2)}(-1)^n[1-\cos(n\pi)], \tag{10}$$

where

$$A = V/(K+S_{1m}). \tag{11}$$

Numerical Simulation

In order to investigate the accuracy of the new approach to HPM solution with a finite number of terms, the system of differential Eqs. (2) and (3) was solved numerically. To show the efficiency of the present method, our results are compared with numerical results graphically. The function pdex4 (Euler's method) in Matlab software which is a function of solving the boundary value problems is used to solve Eqs. (2) and (3) numerically.

Results and Discussion

Equations (7) and (8) represent the new approximate analytical expressions for the substrate and the product concentration profiles for all values of parameters. It satisfies the boundary condition (4). In Fig. 2, we present the

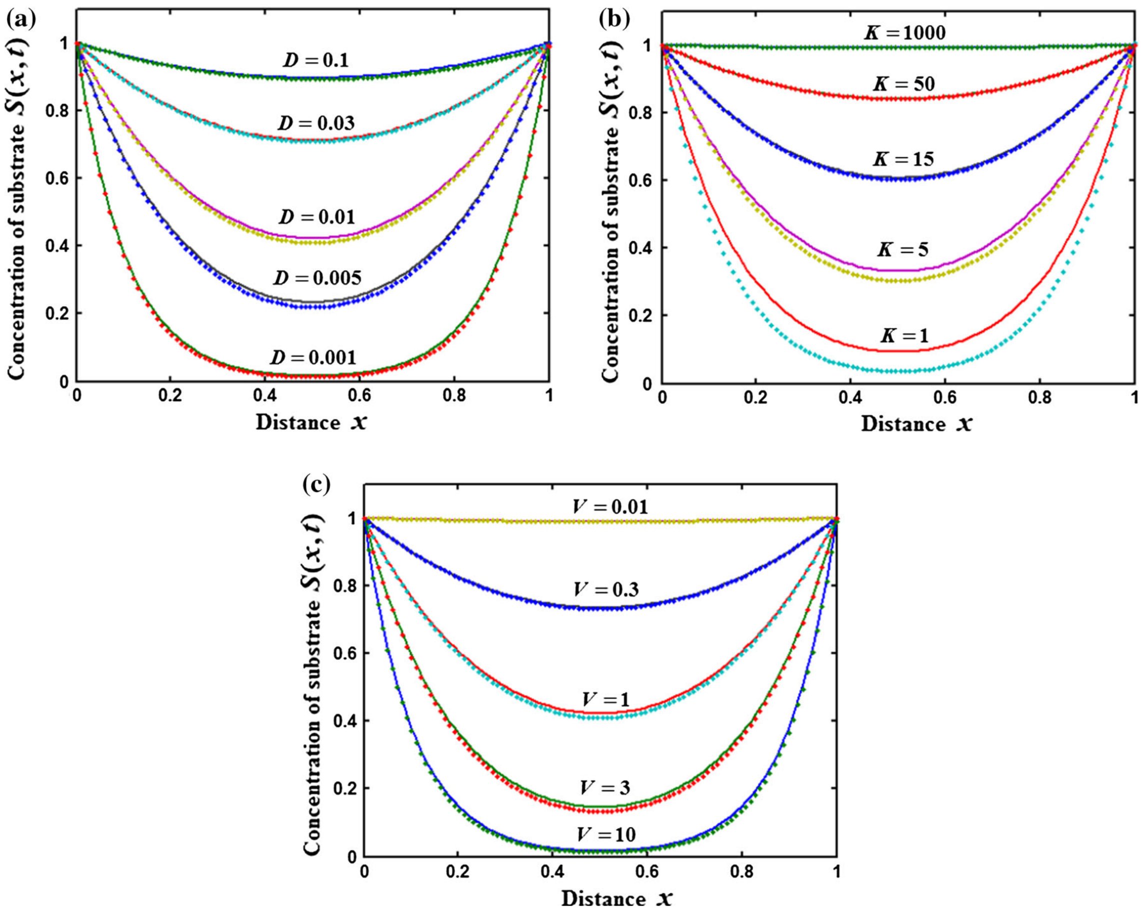

Fig. 2 Plot of concentration of substrate versus distance from the solution membrane interface for various values of **a** Michaelis constant K **b** maximum rate V **c** diffusion coefficient D and for some fixed values of the parameter using Eq. (7). Here the value of $S_i = 1$. *Solid lines* represent the analytical solution obtained in this work; *dotted lines* represent the numerical solution

normalized concentration profiles for a substrate $S\,(x,t)$ as a function of reaction diffusion parameters. From these Figures it is inferred that, the value of S is approximately equal to 1 for small values D and V. From Fig. 2a–c we can also observe that, as the enzymes activity increases, the concentration of substrate falls at the center of the membrane but remains high at the membrane/solution interface, due to diffusion from the bulk and at the membrane/sensor interface due to generation at the membranes. Complete reverse process occurs for the concentration of the product (refer Fig. 3a–c). Flux $J_{S_{1m}}$ versus time for various values of parameter are plotted in Fig. 4. The value of the flux decreases when the value of K,D increases and V decreases. The same process occurs for the flux $J_{P_{1m}}$ (refer Fig. 5).

Conclusions

We have analyzed theoretical model of the non-linear reaction diffusion equations for transport through the liquid and membrane phases. The mass balance equations for substrate and product in immobilized system have been solved analytically using a new approach of HPM. The accuracy of the approximate analytical solutions has been verified by comparison with numerical solutions. The theoretical results obtained can be used for optimization of the performance of the membrane and to improve the design of the membrane.

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Appendix 1: Approximate Analytical Solution of Eq. (2) Using New Approach of Homotopy Perturbation Method

Here, we have indicated how to obtain the solution of Eq. (2) using the initial and boundary condition in Eq. (4). To solve Eq. (2), we first construct the new homotopy approach (Rajendarn and Anitha 2013) as follows:

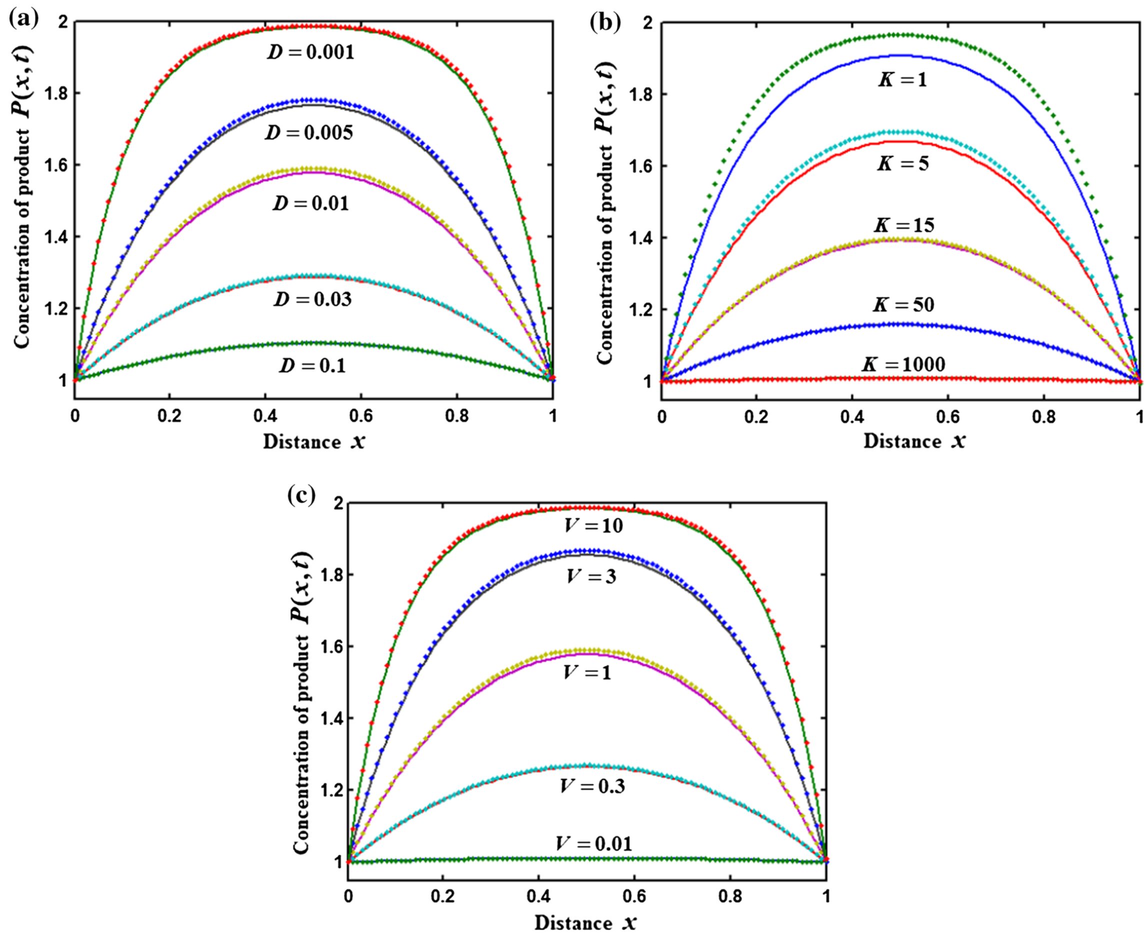

Fig. 3 Plot of concentration of product versus distance from the solution membrane interface for various values of **a** Michaelis constant K **b** maximum rate V **c** diffusion coefficient D and for some fixed values of the parameter using Eq. (8). Here the value of $S_i = 1$. *Solid lines* represent the analytical solution obtained in this work; *dotted lines* represent the numerical solution

$$(1-p)\left[\frac{\partial^2 S(x,t)}{\partial x^2} - \frac{V\,S(x,t)}{K+S(x=0,t)} - \frac{\partial S(x,t)}{\partial t}\right] + p\left[\frac{\partial^2 S(x,t)}{\partial x^2} - \frac{V\,S(x,t)}{K+S(x,t)} - \frac{\partial S(x,t)}{\partial t}\right] = 0 \quad (12)$$

or

$$(1-p)\left[\frac{\partial^2 S(x,t)}{\partial x^2} - \frac{V\,S(x,t)}{K+S_{1m}} - \frac{\partial S(x,t)}{\partial t}\right] + p\left[\frac{\partial^2 S(x,t)}{\partial x^2} - \frac{V\,S(x,t)}{K+S(x,t)} - \frac{\partial S(x,t)}{\partial t}\right] = 0. \quad (13)$$

The approximate solution of Eq. (2) is

$$S = S_0 + pS_1 + p^2 S_2 + \cdots. \quad (14)$$

Substituting the Eq. (14) into Eq. (13) and equating the coefficients of p powers, we get

$$p^0 : \frac{\partial^2 S_0(x,t)}{\partial x^2} - \frac{V\,S_0(x,t)}{K+S_{1m}} - \frac{\partial S_0(x,t)}{\partial t} = 0. \quad (15)$$

The initial and boundary conditions for the above Eq. (15) become

$$\text{At } t = 0,\; S_0 = S_i \quad (16)$$

$$x = 0,\; S_0 = S_{1m} \quad (17)$$

$$x = \bar{x},\; S_0 = S_{2m}. \quad (18)$$

The partial differential Eq. (15) and the corresponding boundary conditions Eqs. (16)–(18) in the Laplace plane become

$$\frac{\partial^2 \bar{S}_0(x)}{\partial x^2} - \frac{V\,\bar{S}_0(x)}{K+S_{1m}} - s\bar{S}_0(x) + S_i = 0. \quad (19)$$

The corresponding boundary conditions are

$$x = 0,\; \bar{S}_0 = S_{1m}/s \quad (20)$$

$$x = \bar{x},\; \bar{S}_0 = S_{2m}/s, \quad (21)$$

where s is the Laplace variable and an over bar indicates a Laplace-transformed quantity. Solving the Eq. (19), and using the boundary conditions and (20) and (21), we can find the following results

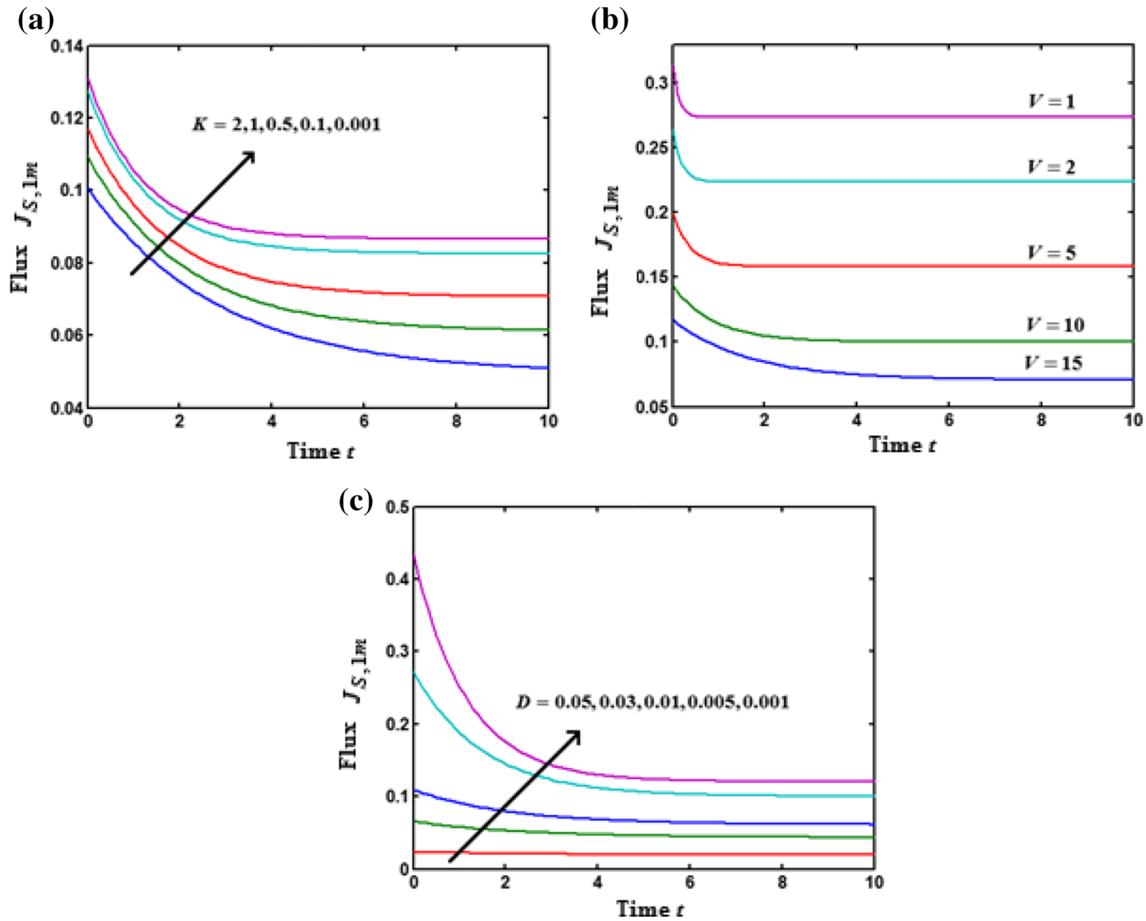

Fig. 4 Plot of flux versus time for various values of **a** Michaelis constant K **b** maximum rate V **c** diffusion coefficient D and for some fixed values of the other parameters using Eq. (9)

$$\bar{S}_0(x) = \frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]} + \frac{S_{2m}\sinh[\sqrt{(A+s)/D}x]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}. \quad (22)$$

Now, we indicate how Eq. (22) can be inverted using the complex inversion formula. If $\bar{y}(s)$ represents the Laplace transform of a function $y(\tau)$, then according to the complex inversion formula we can state that

$$y(\tau) = \frac{1}{2\pi\int_{c-i\infty}^{c+i\infty}\exp[s\tau]\bar{y}(s)\,ds} = \frac{1}{2\pi i}\oint_c \exp[s\tau]\bar{y}(s)\,ds\ , \quad (23)$$

where the integration in Eq. (23) is to be performed along a line $s = c$ in the complex plane where $s = x + iy$. The real number c is chosen such that $s = c$ lies to the right of all the singularities, but is otherwise assumed to be arbitrary. In practice, the integral is evaluated by considering the contour integral presented on the right-hand side of Eq. (23), which is then evaluated using the so-called Bromwich contour. The contour integral is then evaluated using the residue theorem which states for any analytic function $F(z)$.

$$\oint_c F(z)\,dz = 2\pi i\sum_n \mathrm{Re}\,s[F(z)]_{z=z_0}, \quad (24)$$

where the residues are computed at the poles of the function $F(z)$. From Eq. (24), we note that

$$y(\tau) = \sum_n \mathrm{Re}\,s[\exp[s\tau]\bar{y}(s)]_{s=s_0}. \quad (25)$$

From the theory of complex variables, we can show that the residue of a function $F(z)$ at a simple pole at $z = a$ is given by

$$\mathrm{Res}[F(z)]_{z=a} = \lim_{z\to a}\{(z-a)F(z)\}\ . \quad (26)$$

Hence, in order to invert Eq. (22), we need to evaluate

$$\mathrm{Res}\left\{\frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\} + \mathrm{Res}\left\{\frac{S_{2m}\sinh[\sqrt{(A+s)/D}x]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}. \quad (27)$$

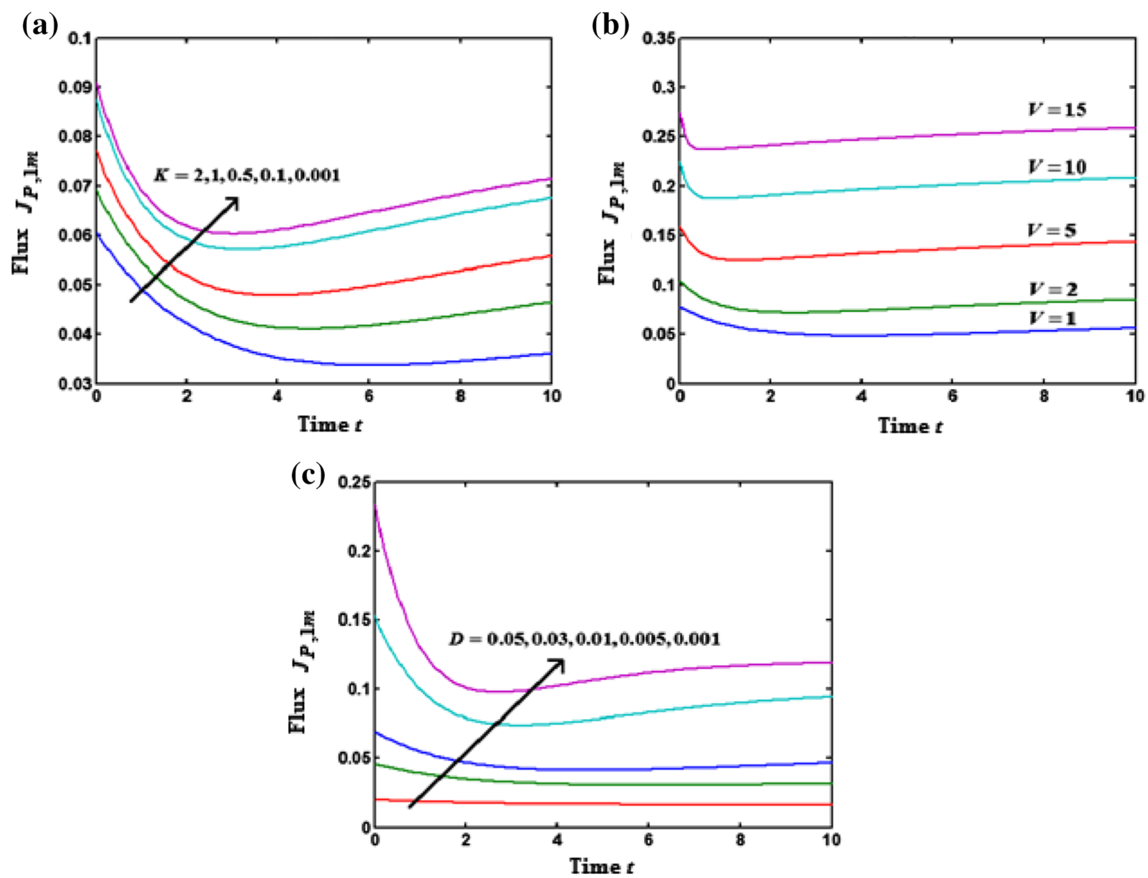

Fig. 5 Plot of flux versus time for various values of **a** Michaelis constant K **b** maximum rate V **c** diffusion coefficient D and for some fixed values of the parameter using Eq. (10)

The poles are obtained from $s\sinh[\sqrt{(A+s)/D}\bar{x}] = 0$. Hence there is a simple pole at $s = 0$, $s = -A$ and there are infinitely many poles given by the solution of the equation $\sinh[\sqrt{(A+s)/D}\bar{x}] = 0$ and so $s_n = \frac{-n^2\pi^2 D}{\bar{x}^2} - A$ where n = 0, 1, 2, … Hence we note that

The residue at $s = 0$ in Eq. (28) is given by

$$\mathrm{Res}\left\{\frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}_{s=0} = \lim_{s\to 0}\left\{\frac{S_{1m}(s-0)e^{st}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\} = \frac{S_{1m}\sinh[\sqrt{A/D}(\bar{x}-x)]}{\sinh[\sqrt{A/D}\bar{x}]}. \tag{29}$$

The residue at $s = s_n$ in Eq. (28) becomes

$$\begin{aligned}&\mathrm{Res}\left\{\frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\} + \mathrm{Res}\left\{\frac{S_{2m}\sinh[\sqrt{(A+s)/D}x]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}\\ &= \mathrm{Res}\left\{\frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}_{s=0} + \mathrm{Res}\left\{\frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}_{s=s_n}\\ &\quad + \mathrm{Res}\left\{\frac{S_{2m}\sinh[\sqrt{(A+s)/D}x]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}_{s=0} + \mathrm{Res}\left\{\frac{S_{2m}\sinh[\sqrt{(A+s)/D}x]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}_{s=s_n}.\end{aligned} \tag{28}$$

$$\begin{aligned}
&\mathrm{Res}\left\{\frac{S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}_{s=s_n}\\
&=\lim_{s\to s_n}\left\{\frac{e^{st}S_{1m}\sinh[\sqrt{(A+s)/D}(\bar{x}-x)]}{s\frac{d}{ds}\sinh[\sqrt{(A+s)/D}\bar{x}]}\right\}\\
&=2\pi D S_{1m}\sum_{n=0}^{\infty}\frac{e^{-(n^2\pi^2D+A\bar{x}^2)t/\bar{x}^2}\sin[n\pi(\bar{x}-x)/\bar{x}]}{(-1)^n(n^2\pi^2D+A\bar{x}^2)}
\end{aligned} \tag{30}$$

Similarly obtaining the residues of other terms in Eq. (28), finally we get Eq. (7) in the text.

Appendix 2: Approximate Analytical Solution of Eq. (3) Using the Relation between Substrate and Product Concentrations

Adding Eqs. (1) and (2) we get

$$\frac{\partial S}{\partial t}+\frac{\partial P}{\partial t}=D_s\frac{\partial^2 S}{\partial x^2}+D_P\frac{\partial^2 P}{\partial x^2} \tag{31}$$

$$\frac{\partial}{\partial t}(S+P) = \frac{\partial^2}{\partial x^2}(SD_s+PD_p) \tag{32}$$

Let us take $D_s = D_p = D$ and $G = S + P$, we get (33)

$$\frac{\partial G}{\partial t}=D\frac{\partial^2 G}{\partial x^2}. \tag{34}$$

The boundary conditions for the above equation become

$$\begin{aligned}
&t=0,\ G=S_i\\
&x=0,\ G=S_{1m}+P_{1m}\ .\\
&x=\bar{x},\ G=S_{2m}+P_{2m}
\end{aligned} \tag{35}$$

The solution of the above equations is

$$\begin{aligned}
G&=(S_{1m}+P_{1m})+[(S_{2m}+P_{2m})-(S_{1m}+P_{1m})]\frac{x}{\bar{x}}\\
&+\frac{2}{\pi}\sum\left[\frac{(S_{2m}+P_{2m})\cos n\pi-(S_{1m}-P_{1m})}{n}\right]\\
&\times\sin\left(\frac{n\pi x}{\bar{x}}\right)e^{\frac{-Dn^2\pi^2t}{\bar{x}^2}}\\
&+2S_i\sum_{n=0}^{\infty}\frac{[1-(-1)^n]}{n\pi}\sin\left(\frac{n\pi x}{\bar{x}}\right)\ e^{\frac{-Dn^2\pi^2t}{\bar{x}^2}}
\end{aligned} \tag{36}$$

Using the relation $G = S + P$, we obtain the Eq. (8) in the text.

Appendix 3: Matlab Program to Find the Numerical Solution of Eqs. (3) and (4)

```
function see5
m =0;
x =linspace(0,1);
%x=[0 0.2 0.4 0.6 0.8 1];
t=linspace(0,1000000);
sol=pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
u2 = sol(:,:,2);
figure
%surf(x,t,u1);
plot(x,u1(end,:))
title('u1(x,t)')
xlabel('Distance x')
ylabel('time ')
%-----------------------------------------------------------------
figure
plot(x,u2(end,:))
title('u2(x,t)')
xlabel('Distance x')
ylabel('u2(x,2)')
%-----------------------------------------------------------------
function [c,f,s] = pdex4pde(x,t,u,DuDx)
D=0.01;
c =[1;1];
f =[D;D].*DuDx;
v =0.75;k=1000;
F=-v*u(1)/(k+u(1));
F1=v*u(1)/(k+u(1));
s=[F,F1];
% --------------------------------------------------------------
function u0 = pdex4ic(x)
u0 =[1;0];
% --------------------------------------------------------------
function [pl,ql,pr,qr]=pdex4bc(xl,ul,xr,ur,t)
s1m=1;s2m=1;p1m=1;p2m=1;
pl = [ul(1)-s1m;ul(2)-p1m];
ql = [0;0];
pr = [ur(1)-s2m;ur(2)-p2m];
qr = [0;0];
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